Unit-3

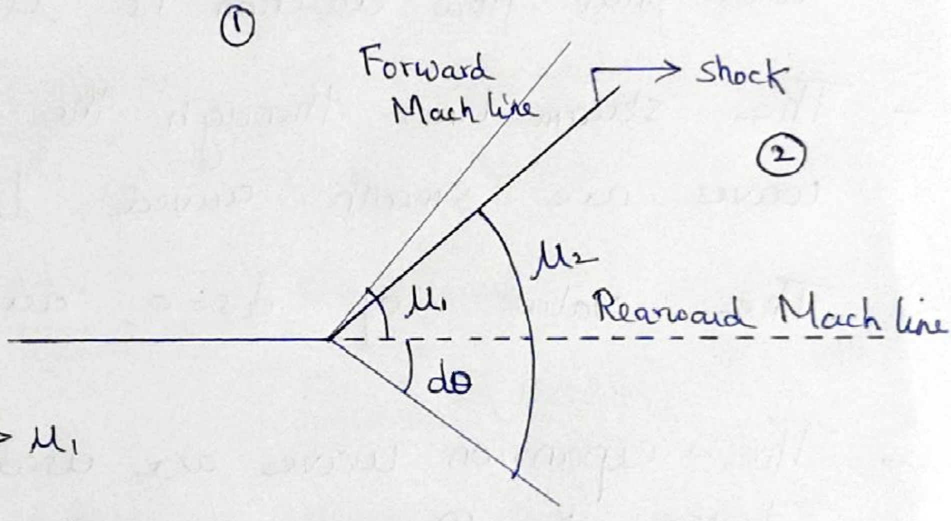
Expansion Waves and Method of Characteristics

Expansion waves :-

If oblique shocks occurs at the Corner, expansion wave is formed.

Prandtl - Meyer function for expansion wave :-

①



(i) $\mu_2 > \mu_1$

(ii) $\frac{P_2}{P_1} < 1$

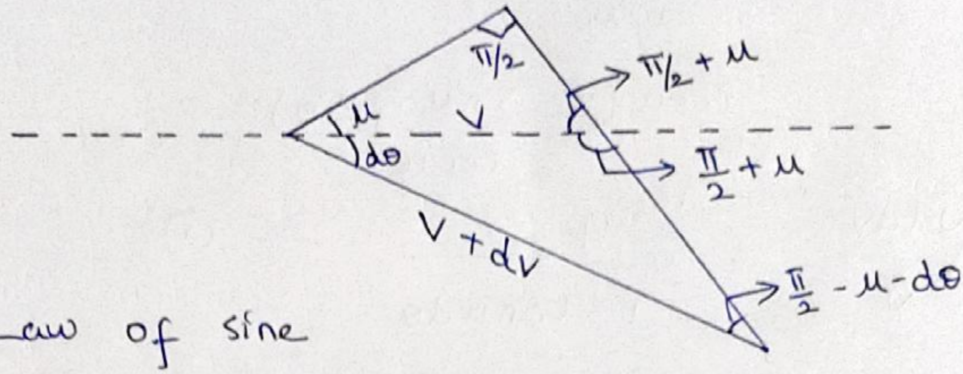
(iii) $\frac{T_2}{T_1} < 1$

(iv) $\frac{\rho_2}{\rho_1} < 1$

$M_1 > 1$

$\theta = 0$

- When the flow is turned away from itself expansion wave is formed
- Expansion wave consist of infinite number of mach waves
- The angle between forward mach line and initial flow direction is μ_1
- The angle between rearward machline and final flow direction is μ_2
- The stream lines through the expansion waves are smooth curved lines.
- The value of $ds = 0$, across the wave.
- The expansion waves are used to identify the Prandtl - Meyer function. So, this wave is also known as Prandtl - Meyer expansion wave.



Law of sine

$$\frac{\sin\alpha_1}{A} = \frac{\sin\alpha_2}{B} = \frac{\sin\alpha_3}{C}$$

$$\frac{\sin\left(\frac{\pi}{2} - \mu - d\theta\right)}{V} = \frac{\sin\left(\frac{\pi}{2} + \mu\right)}{V + dV}$$

$$\frac{V+dV}{V} = \frac{\sin\left(\frac{\pi}{2} + \mu\right)}{\sin\left(\frac{\pi}{2} - (\mu + d\theta)\right)}$$

$$\frac{V+dV}{V} = \frac{\cos\mu}{\cos(\mu + d\theta)}$$

$$\frac{V+dV}{V} = \frac{\cos\mu}{\cos\mu\cos d\theta - \sin\mu\sin d\theta}$$

$\cos d\theta = 1$

$\sin d\theta = d\theta$

$$\frac{v+dv}{v} = \frac{\cos\mu}{\cos\mu - \sin\mu\, d\theta}$$

$$= \frac{\cancel{\cos}\mu}{\cancel{\cos}\mu\left(1 - \frac{\sin\mu}{\cos\mu}\, d\theta\right)}$$

$$\frac{v+dv}{v} = \frac{1}{1-\tan\mu\, d\theta} \quad — ①$$

$$\frac{v+dv}{v} = (1-\tan\mu\, d\theta)^{-1}$$

Using Binomial function

$$(1+x)^{-1} = 1+x+x^2+x^3+\ldots$$

$$(1-\tan\mu\, d\theta)^{-1} = 1+\tan\mu\, d\theta$$

$$\frac{v+dv}{v} = 1+\tan\mu\, d\theta$$

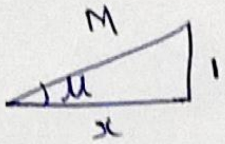
$$\cancel{1}+\frac{dv}{v} = \cancel{1}+\tan\mu\, d\theta$$

$$\frac{dv}{v} = \tan\mu\, d\theta$$

$$d\theta = \frac{dv}{v\tan\mu} \quad — ②$$

Mach angle

$$\sin \mu = \frac{1}{M}$$

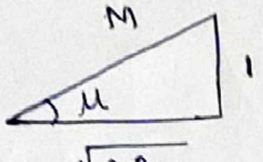


By Pythogoras theorem

$$M^2 = x^2 + 1$$

$$x^2 = M^2 - 1$$

$$x = \sqrt{M^2 - 1}$$



$$\tan \mu = \frac{1}{\sqrt{M^2 - 1}} \quad -(3)$$

(3) in (2)

$$d\theta = \frac{dV}{V \times \frac{1}{\sqrt{M^2 - 1}}}$$

$$d\theta = \frac{dV}{V} \times \sqrt{M^2 - 1} \quad -(4)$$

Eqn (4) is known as governing eqn for Prandtl - Meyer function.

$$M = \frac{V}{a}$$

$$V = Ma \quad — ⑤$$

$$\ln V = \ln (Ma)$$

$$\ln V = \ln M + \ln a \rightarrow ⑥$$

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} \quad — ⑦$$

WKT

$$\frac{T_0}{T} = 1 + \frac{(\gamma - 1)}{2} M^2$$

$$a^2 = \gamma R T$$

$$a_0^2 = \gamma R T_0$$

$$\frac{a_0^2}{a^2} = \frac{\cancel{\gamma} R T_0}{\cancel{\gamma} R T}$$

$$\frac{a_0}{a} = \sqrt{\frac{T_0}{T}}$$

$$a = \frac{a_0}{\sqrt{\frac{T_0}{T}}}$$

$$a = \frac{a_0}{\sqrt{1 + \frac{(\gamma - 1)}{2} M^2}}$$

$$\ln a = \ln\left[\frac{a_0}{\left(1+\frac{(\gamma-1)}{2}M^2\right)^{1/2}}\right]$$

$$\ln a = \ln a_0 - \ln\left(1+\frac{(\gamma-1)}{2}M^2\right)^{1/2}$$

$$\ln a = 0 - \frac{1}{2}\ln\left[1+\frac{(\gamma-1)}{2}M^2\right] \longleftarrow (8)$$

$$\frac{da}{a} = -\frac{1}{\cancel{2}} \times \frac{1}{\left(1+\frac{(\gamma-1)}{2}M^2\right)} \frac{(\gamma-1)}{2} \cancel{2} M\,dM$$

$$\frac{da}{a} = \frac{-\frac{(\gamma-1)}{2}M\,dM}{1+\frac{(\gamma-1)}{2}M^2} \quad \text{---} (9)$$

From (7)

$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a}$$

(9) in (7)

$$\frac{dv}{v} = \frac{dM}{M} - \left[\frac{\frac{(\gamma-1)}{2}M\,dM}{1+\frac{(\gamma-1)}{2}M^2}\right] \quad \text{---} (10)$$

From (4)

$$d\theta = \frac{dV}{V}\sqrt{M^2-1}$$

(10) in (4)

$$d\theta = \left\{\frac{dM}{M} - \left[\frac{\frac{(\gamma-1)}{2} M\,dM}{1+\frac{(\gamma-1)}{2}M^2}\right]\right\}\sqrt{M^2-1}$$

$$= \frac{dM}{M}\left[1 - \frac{\left(\frac{(\gamma-1)}{2} M\,\cancel{dM} \times \frac{M}{\cancel{dM}}\right)}{1+\frac{(\gamma-1)}{2}\cdot M^2}\right]\sqrt{M^2-1}$$

$$d\theta = \frac{dM}{M}\left[1 - \frac{\left(\frac{(\gamma-1)}{2} M^2\right)}{1+\frac{(\gamma-1)}{2} M^2}\right]\sqrt{M^2-1}$$

$$= \frac{dM}{M}\left[\frac{1+\cancel{\frac{(\gamma-1)}{2}M^2} - \cancel{\frac{(\gamma-1)}{2}M^2}}{1+\frac{(\gamma-1)}{2} M^2}\right]\sqrt{M^2-1}$$

$$d\theta = \frac{dM}{M}\left[\frac{\sqrt{M^2-1}}{1+\frac{(\gamma-1)}{2} M^2}\right] \quad \text{— (11)}$$

Eqn (12) is known as flow deflection angle

$$\theta = \int \frac{dM}{M}\left[\frac{\sqrt{M^2-1}}{1+\frac{(\gamma-1)}{2} M^2}\right] \longrightarrow (12)$$